ON MULTIFORM FUNCTIONS DEFINED BY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

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Addressing the third international mathematical congress, in 1904, Painlevé stated that:* Generally speaking, an irreducible differential equation should be regarded as integrated or solved if, by any analytical process, we are able to represent the general solution in its whole field of existence with any required approximation, the representation displaying the fundamental properties of the equation, showing how the initial conditions enter in, etc.

I wish to suggest a method which, from such a viewpoint, may be said to solve a number of differential equations of the first order. In addition, this method may be of some interest as introducing certain multiform functions (associated with the solutions of the differential equations) which have remarkable automorphic properties,

As the preliminary studies on which the method is based, and the discussion of the numerous cases which it involves, cannot be exposed without entering into long developments, it may not be useless to state the main results† separately and illustrate them by an example, in the brief summary which I am presenting here. A longer article on the same subject will be printed in the Annales de l'Ecole Normale Supérieure.

1. **Classification of a family of differential equations.** The only equation of the first order and degree which defines uniform functions is, as is well known, the Riccati equation. I have made repeated efforts to investigate another equation which, for many reasons, seems to be next in simplicity, that is

$$y' + A_0 + A_1 y + A_2 y^2 + A_3 y^3 = 0, \tag{1}$$

where the A's are *rational* functions of x, and, first of all, polynomials. The study of this equation, which I shall write hereafter

$$zz' = A_0 z^3 + A_1 z^2 + A_2 z + A_3; \qquad z = y^{-1}, \tag{2}$$

leads to following classification:

First Case. Let $A_0 = A_1 = 0$ identically. Then, in the neighborhood of $x = \infty$, the single *branches* of the solutions of (2) are of the *algebroid*

* Verhandl. des III. Intern. Math. Congr., Heidelberg, p. 96.

† Some of these results have been given in short papers published in the Comptes rendus de l'Académie des Sciences, Paris.

type. I mean by this that, if we follow *on all straight lines starting from any point* a solution $z(x)$ of (2) (thus obtaining a *branch* of this solution), there will exist an algebraic function $Z(x)$ such that $z(x)/Z(x)$ approaches 1 when x approaches ∞. (In other words the branch is asymptotic to an algebraic function).

Second Case. Let A_0 alone be identically equal to 0. Then, in the neighborhood of $x = \infty$, the branches $z(x)$ are of the *exponential type;* they behave like exponential functions; in particular, no one of them ever becomes infinite, but presents an infinite number of *roots* (which, for it, are singular points, since equation (2) offers a singularity wherever $z = 0$).

Third Case. A_0 and A_1 are *not* 0. Then the branches $z(x)$ are, in the vicinity of $x = \infty$, of the *meromorphic type;* for each of them we have an infinite number of roots and an infinite number of infinites.

In view of this, it will be natural to leave cases 2 and 3 for a later investigation and to concentrate our efforts on Case 1.

Now, in this case, calling m_2 and m_3 the degrees of A_2 and A_3 in x, I have been led to make further distinctions. I say that equation (3),

$$zz' = A_2 z + A_3, \tag{3}$$

is of *type A* when $m_3 > 2m_2 + 1$,
of *type B* when $m_3 < 2m_2 + 1$,
of *type C* when $m_3 = 2m_2 + 1$.

The type which I am going to consider at present is *type A*.

2. **A simple equation of type *A* here proposed as an example.** To make as clear as possible the facts which I wish to submit, I shall limit my statements, in this summary, to one of the simplest examples of equations of type A which can be formed. This is an equation (3) where $m_3 = 3$, $m_2 = 0$, namely

$$zz' = 3mz + 2(x^3 - 1) \qquad (m \text{ constant}). \tag{4}$$

To discuss this equation fully, I shall assume later that m is a real negative number of small absolute value.

3. **First properties of the solutions of** (4). The solutions $z(x)$ offer no *poles,* nor other points where they are infinite. Their only *algebraic singular points* are their roots (points where $z = 0$), each of which exchanges two determinations of the vanishing branch. The only *transcendental* singular points of the equation are

$$x = \alpha = e^{2i\pi/3}; \; x = \beta = e^{-(2i\pi/3)}; \quad x = \gamma = 1, \qquad \text{for special branches,}$$

$$x = \infty, \qquad \text{for all branches.}$$

Let us call $\bar{x}$ a point which we make approach ∞ on the negative

real axis; call $2y$ the value of $z^2 - \bar{x}^4 + 4\bar{x}$ at this point, and follow any path from $\bar{x}$ to any point x. If $z(x)$ is a solution of (4), we have in x:

$$Z = z^2 = x^4 - 4x + 2y + 6m \int_{\infty}^{x} z dx, \tag{5}$$

the integral being taken along the appointed path from $\bar{x}$ (removed to ∞) to x.

Let us consider such branches $Z(x)$ which are obtained by starting from $\bar{x}$ (that is ∞) with a given y and following the set of all straight lines parallel to the real axis. Such a branch is, in the neighborhood of $x = \infty$, asymptotic to the polynomial $x^4 - 4x + 2y$; furthermore, if $|y|$ is large, the ratio of its value to $x^4 - 4x + 2y$ for any x on the straight lines described is close to 1 and approaches 1 when y approaches ∞. It follows that, if $|y|$ is large, the said branch offers exactly 4 roots which are respectively in the vicinities of the roots of $x^4 - 4x + 2y$. (This fact which is always true for any given equation (4) as soon as $|y|$ is above a certain numerical function of the coefficient m in the equation, would be true for all values of y if we gave to m a value approaching 0).

As $Z = z^2$, the branches $z(x)$ corresponding to the branches $Z(x)$ just described are asymptotic to $\sqrt{x^4 - 4x + 2y}$. There are, therefore, two sets of branches $z(x)$, the ones becoming infinite like $+ x^2$ (on the straight lines followed) and the others like $- x^2$.

Each of the branches Z or z so defined may be represented in the whole x-plane by a limited number of Taylor's series; that is obvious for a large $|y|$ from what has just been said, and it can be proved that it is always so. Let us add that there exist, in fact, other much more convenient single expansions for these functions (or rather branches); but I shall not enter into this question here, and will only point out that, as soon as a value of y is given, we are able to represent the corresponding branch wholly and to determine its behavior. The only problem left therefore, in order to know any one of the multiform functions in its whole field of existence is to determine the whole set of values of y which belong to the same function (that is, which define branches $z(x)$ all belonging to the same multiform function).

4. **Another definition of parameter** y. In order to understand the part played in the question by parameter y, it is useful to show its connection with another parameter which may be defined as follows:

In the neighborhood of $x = \infty$, as we have seen, there are two sets of branches $z(x)$. Let us consider those branches that become infinite like $+ x^2$. They may be represented by the expansion

(6) $z = x^2 + mx + m^2 +$ *expansion in powers of* x^{-1} and *of* $(C_1 + \eta_1 \log x)x^{-2}$,

where $\eta_1 = 6m(m^3 - 1)$, while C_1 is a parameter.

The set of branches becoming infinite like $-x^2$ may be represented by a similar expansion $z = -x^2 + mx + \textit{terms in } x^{-1} \textit{ and } (C_2 + \eta_2 \log x)x^{-2}$, where $\eta_2 = 6m(m^3 + 1)$ and C_2 is another parameter.

Let us decide always to give to $\log x$ in (6) *such values as have their imaginary part between* 0 *and* $2i\pi$: there is, then, a one-to-one correspondence between the different values of C_1 and the different branches z defined by (6). Similarly, we define a one-to-one correspondence between the values of C_2 and the branches of the second set.

According to these assumptions, there will correspond to any individual multiform function $z(x)$: (1) a set of values of C_1; (2) a set of values of C_2. We shall, hereafter, fix our attention on the C_1 only. It is obvious that whatever we are doing with C_1 could be done with C_2 also.

To introduce the parameter C_1 is, in fact, to give a new definition of y, for we have (taking the square of the right hand member of (6))

$$y = -2m + \frac{5}{2}m^4 + C. \tag{7}$$

5. **The group problem.** We may now state the question before us as follows:

Starting from $\bar{x}$ removed to ∞ with z equal to a branch of the first set (that is becoming infinite like $+x^2$ and corresponding to a value of C_1) let us follow z along any possible closed path which brings us back in $\bar{x}$ with z equal to a branch of the same set. If the two values of C_1 corresponding to the beginning and the end of the path are different, we may say that the path *exchanges* those two values of C_1 or the corresponding values of y (defined by (7)). In other words, the closed path operates a *substitution* (S) on y, say $[y_0, y_1]$. Our problem is to find all possible values y_1 which may be so exchanged with the same y_0. In other words, we have to determine all the substitutions (S).

It follows from formula (5), that if we call Γ any of the closed paths interchanging two values of y, the substitutions before us are the substitutions $[y,\ y + 3m\int_\Gamma z\,dx]$. And, knowing (see 3) that z is asymptotic to the square root of a polynomial of the fourth degree, we may foresee that the study of the substitution (S) will not be without analogy with that of elliptic integrals. The condition, for instance, that all values of y considered in the substitutions should correspond to values of C_1 is exactly the condition which is needed in order to define the periods of elliptic integrals: only such closed paths, namely, can define periods of the integral $\int \sqrt{x^4 - 4x + y}\,dx$, on which $\sqrt{x^4 - 4x + y}$ has the same sign at the initial and final point, that is to say such paths as effect an *even* number of single permutations.

The substitutions (S) satisfying the conditions prescribed evidently form a discontinuous group: it is this group—group (G)—that we have to investigate.

6. **Introducing the ψ-functions.** The chief question concerning group (G) is obviously the following: Is it possible to build up that group with a limited number of simple fundamental substitutions? To answer this question, we shall introduce new multiform functions, which I call ψ, and which, for the special equation which I have here in view, may be defined as follows:

If, in equation (5), we make $m = 0$ and $y = 0$, we have $z = \sqrt{x^4 - 4x}$. The function z then has only 4 roots (or critical points) which are $x_1 = 0$, $x_2 = e^{-(2i\pi/3)} \sqrt[3]{4}$, $x_3 = \sqrt[3]{4}$, $x_4 = e^{2i\pi/3} \sqrt[3]{4}$. From $\bar{x}$ (removed to ∞ in the direction of the negative real axis, see 3), we draw a closed negatively sensed circuit Γ_1 surrounding x_1 and x_2, and another closed positively

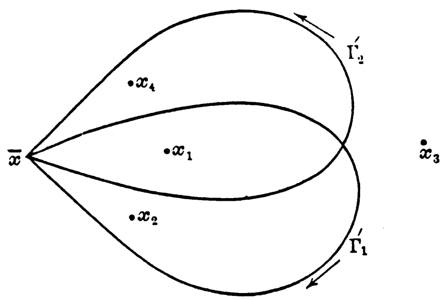

FIG. 1.

sensed circuit Γ_2 surrounding x_1 and x_2. For small values of y and m, the $z(x)$ defined by (5) (the one which in $\bar{x}$ becomes infinite like $+\bar{x}^2$) will be holomorphic all along Γ_1 and Γ_2: therefore the equalities

$$(8) \qquad \psi_1(y) = y + 3m \int_{\Gamma_1} z d\bar{x}; \qquad \psi_2(y) = y + 3m \int_{\Gamma_2} z dx$$

define functions of y which, for any small $|m|$, are holomorphic in the neighborhood of $y = 0$. Supposing now that m and y vary, let us follow the variation of the two functions (8). In order to do that, we shall be obliged, as $|y|$ or $|m|$ grows, to alter the figure of the curves Γ_1 and Γ_2; we have namely to submit these curves to a continuous deformation which makes it possible to avoid encountering on them any critical point of the branch z integrated (in the right-hand members of (8)). As long as such a continuous deformation is possible, the functions $\psi_1(y)$ and $\psi_2(y)$ cannot cease to be holomorphic. But in only one case will the deformation become impossible: namely when the curve Γ_1 or Γ_2 has to pass *between*

two critical points of z which come to coincide; in this last case, Γ_1 or Γ_2 *must* cross the multiple singular point, which is bound to be one of the three transcendent singularities of the differential equation, namely

$$x = \alpha = e^{2i\pi/3}; \qquad x = \beta = e^{-(2i\pi/3)}; \qquad x = \gamma = 1 \qquad \text{(see 3).}$$

In order to make the full discussion which follows geometrically simple, we shall assume, henceforth, that m_1 in equations (4) and (5), is *a negative real number of small absolute value* (see 2). This value of m is now supposed to be fixed while we move y on all straight lines starting from $y = 0$. On this set of lines of the y-plane, we get *one* special branch of $\psi_1(y)$ and *one* special branch of $\psi_2(y)$ on which we shall first and *provisionally* fix our attention.

Under these conditions, I prove that, following the two ψ's on the said straight lines, I find only:

Two singularities of $\psi_1(y)$, namely one, y_α, for which Γ_1 crosses $x = \alpha$, and one, y_γ, for which Γ_1 crosses $x = \gamma$;

Two singularities of $\psi_2(y)$, namely one, y_β, for which Γ_2 crosses $x = \beta$, and the other, y_γ (the same as y_γ as above), for which Γ_2 crosses $x = \gamma$.

To these singularities correspond:

Two singularities of $\psi_1^{(-1)}(y)$, the inverse function of ψ_1, namely $y_{\alpha 1}$ and $y_{\gamma 1}$ such that $y_{\alpha 1} = \psi_1(y_\alpha)$, $y_{\gamma 1} = \psi_1(y_\gamma)$;

Two singularities of $\psi_2^{(-1)}(y)$, the inverse function of ψ_2, namely $y_{\beta 2} = y_2(y_\beta)$ and $y_{\gamma 2} = \psi_2(y_\gamma)$.

As for the position of the points y_α, etc., we easily see, that, if $|m|$ is small (as assumed), y_α and $y_{\alpha 1}$ are close to the value $3/2e^{2i\pi/3}$ which they have for $m = 0$; y_β and $y_{\beta 2}$ are close to $3/2e^{-(2i\pi/3)}$; y_γ, $y_{\gamma 1}$ and y_γ^2 are close to $3/2$; furthermore, if m is real and negative, y_γ is real and positive while the couples y_α and y_β, $y_{\alpha 1}$ and $y_{\beta 2}$, $y_{\gamma 1}$ and $y_{\gamma 2}$ are conjugate imaginaries, y_α, $y_{\alpha 1}$ and $y_{\gamma 2}$ being above the real x-axis.

Besides, whatever be m, we have

$$y_{\gamma 2} - y_{\gamma 1} = 2i\pi\eta_1,$$

η_1 being the number defined in section 3 above (see expansion (6)).

7. **Definition of fields $F_1, \ldots F_2$ and of a set of fundamental substitutions (S).** Let us draw and name the following lines in the y-plane:

$0y_\gamma\infty$, real positive half-axis from $y = 0$ to $y = \infty$;

$0_1y_{\gamma 1}\infty$ and $0_2y_{\gamma 2}\infty$, transformed of $0y_\gamma\infty$ by the substitutions $[y, \psi_1(y)]$, $[y, \psi_2(y)]$, *where we take for ψ_1 and ψ_2 the branches of the ψ-functions previously defined in* 6.

$0y_\alpha$ and $0y_\beta$ straight lines;

$0_1y_{\alpha 1}$ and 0_1y_β transformed of $0y_\alpha$ by the substitution $[y, \psi_1]$ and $[y, \psi_2]$ defined as above;

0_2y_α and $0_2y_{\beta 2}$, transformed of $0y_\beta$ by the same substitutions.

The lines so defined are shown in Fig. 2. If m is real, negative and small (as assumed), all these lines are approximately straight (when not exactly); the line $y_\alpha 0_2 y_{\gamma 2}\ \infty$ is above the line $y_\alpha 0 y_\gamma\ \infty$ and does not cut that line, nor cut itself; furthermore $y_\alpha 0_2 y_{\gamma 2}\ \infty$ and $y_\beta 0_1 y_{\gamma 1}\ \infty$ are symmetrical with respect to the real axis; so are also $0_1 y_{\alpha 1}$ and $0_2 y_{\beta 2}$.

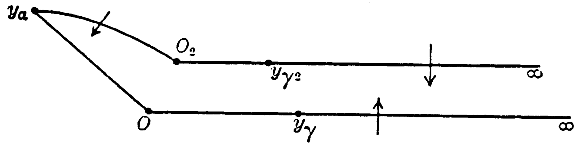

FIG. 2.

I shall now call $F_1, \ldots F_2$ the fields in the y-plane thus defined:

F_1 is the outside of the closed curve $\infty\ y_\gamma 0 y_\alpha 0_2 y_{\gamma 2}\ \infty$;

F_2 is the outside of the closed curve $\infty\ y_\gamma 0 y_\beta 0_1 y_{\gamma 1}\ \infty$;

F_{-1} is the whole y-plane *cut* by the line $y_{\alpha 1} 0_1 y_{\gamma 1}\ \infty$;

F_{-2} is the whole y-plane *cut* by the line $y_{\beta 2} 0_2 y_{\gamma 2}\ \infty$.

In the respective fields $F_1, \ldots F_{-2}$, let us consider the functions ψ_1, ψ_2, and the inverse functions $\psi_1^{(-1)}$, $\psi_2^{(-1)}$, and, more precisely those determinations of the four functions which are such that $\psi_1(y_\alpha) = y_{\alpha 1}$, $\psi_2(y_\beta) = y_{\beta 2}$; $\psi_1^{(-1)}(y_{\alpha 1}) = y_\alpha$; $\psi_2^{(-1)}(y_{\beta 2}) = y_\beta$. Of each of the four functions I have one single uniform and holomorphic branch or *element* in the corresponding field. Let us call $\hat{\psi}_1, \ldots \hat{\psi}^{(-2)}$ the uniform elements of functions so defined (it will be noticed that, according to the definition of the fields F, $\hat{\psi}_1^{(-1)}$ and $\hat{\psi}_2^{(-2)}$ are defined for all values of y, while $\hat{\psi}_1$ and $\hat{\psi}_2$ are not).

The elements of functions $\hat{\psi}_1 \ldots \hat{\psi}_2^{(-2)}$ are analytically defined by equations (8) and, being uniform, may be easily represented by convergent expansions in their respective fields. From the point of view of Painlevé, they are "known functions."

We have to remark that fields the F_1 and F_{-1} (or F_2 and F_2^{-1}) do not correspond exactly to each other: in other words, when y describes the boundary of F_1, the corresponding point $\hat{\psi}_1(y)$ does not describe the boundary of F_{-1} from end to end. If, however, we superpose two y-planes one bearing F_1 and F_2 and the other F_{-2} and F_{-1} and consider in the two sheets two regions, composed, one of F_1 and F_{-2}, the other of F_{-1} and F_2, we have an exact correspondence between these two regions. In other words, let us call $\mathcal{F}_1$, on the two-sheets, the field bounded by the line $\infty\ y_\gamma 0 y_\alpha 0_2$ (*first sheet*) $0_2 y_{\beta 2} 0_2 y_{\gamma 2}\ \infty$ (*second sheet*); let us call $\mathcal{F}_2$ the field bounded by the line $\infty\ y_\gamma 0 y_\beta 0_1$ (*first sheet*), $0_1 y_{\alpha 1} 0_1 y_{\gamma 1}\ \infty$ (*second sheet*). When the point y, starting from y_α (first sheet) covers the whole area $\mathcal{F}_1$, the corresponding

point $\psi_1(y)$ starting from $y_{\alpha 1}$ (second sheet) covers the whole area $\mathcal{F}_2$ (here, of course, $\psi_1(y)$ coincides with $\hat{\psi}_1(y)$ in one part of $\mathcal{F}_1$ only; in the remaining part, it coincides with $\hat{\psi}_2^{(-1)}(y)$, as may be deduced from the facts stated in section 9 below).

8. **Solution of the group-problem.** The "elements of functions" $\hat{\psi}_1(y)$, $\hat{\psi}_2(y)$, ... $\hat{\psi}_2^{(-1)}(y)$ being defined as stated in 7 (in their respective fields), let us call (S_1), ... (S_2^{-1}) the substitutions $(S_1) = [y, \hat{\psi}_1(y)]$, $(S_2) = [y, \hat{\psi}_2(y)]$, ... $(S_2^{-1}) = [y, \hat{\psi}_2^{(-1)}(y)]$. To these substitutions and inverse substitutions we add a third

$$(S_3) = [y, y + 2i\pi\eta_1], \qquad (S_3^{-1}) = [y, y - 2i\pi\eta_1]$$

which is uniformly defined for every y (η_1 is the number defined in 4). This being done, I come to the following result which I will submit here without proof: *All substitutions* (S) *of group* G *(see* 5*) are products of the three substitutions* (S_1), (S_2), (S_3) *and the inverse substitutions.*

In other words, *with the three fundamental substitutions* (S_1), (S_2), (S_3) *(and inverse substitutions) defined by uniform elements of* ψ*-functions we can build up the total group* G.

Reciprocally, of course, any product of (S_1), ... (S_3^{-1}) is a substitution of G. But we have to remember the fact that for some values of y it will happen that one of the fundamental substitutions will cease to exist. Namely (S_1) is *not* defined for y inside of the curve $\infty\, 0 y_\gamma y_\alpha y_{\gamma 3}\, \infty$ (see Figure 2), and (S_2) is *not* defined for y inside of the curve $\infty\, 0 y_\beta y_\gamma\, \infty$.

The multiplication of substitution (S_1), ... (S_3^{-1}), it will be noticed, is *not commutative*.

9. **Automorphic properties of the ψ-functions.** The ψ-functions are multiform functions which have the property that the knowledge of two branches or "elements" of these functions is sufficient to get the whole functions (in their whole field of existence) by means of multiplication of substitutions. In other words, any branch of a ψ-function is made up of a few fundamental branches combined together. This character may be described as an automorphic property of the function.

In order to verify that the ψ-functions actually have this property, we have to show this: The initial elements of the ψ's have been defined in certain fields F_1, ... F_2^{-1} of the y-plane; we have to show that whenever we cross a boundary of F_1, F_2, ... or F_2^{-1}, the corresponding substitution (S_1), (S_2), ... (S_2^{-1}), which then ceases to be defined as one of the fundamental substitutions, *will be transformed into some product of the fundamental substitutions.*

Let us, for instance, state here what becomes of (S_1) when we cross the boundaries of F_1. I prove that:

if we cross the line $y_\gamma\ \infty$ (coming from the inside of F_1, that is from below on the diagram), $(S)_1$ is changed for $(S_2 . S_3^{-1})$;

if we cross the line $y_\gamma 0 y_\alpha$, (S_1) is changed into $(S_2 . S_1)$;

if we cross the line $y_\alpha 0_2$ (coming from F_1), (S_1) is changed into $(S_2^{-1} . S_1)$;

if we cross the line $0_2 y_{\alpha 2}\ \infty$, (S_1) is changed into (S_2^{-1}).

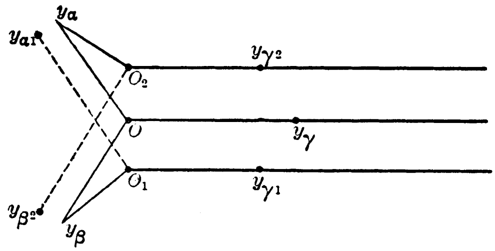

FIG. 3.

All the above combinations of substitutions are *defined* (*exist*) for the value of y to which they are applied. For instance, let us consider $(S_2 . S_1)$ in the points y which are met when we cross $y_\gamma 0 y_\alpha$, that is just above the line $y_\gamma 0 y_\alpha$ on the diagram: (S_2) operating on such points carries us *above* the line $y_\alpha 0_2 y_{\gamma 2}$, where (S_1) is defined.

It will be noticed, that, considered in their whole field of existence, the four functions $\psi_1, \ldots \psi_2^{(-1)}$ consist of only one function and the inverse function, as follows from the fact that ψ_1 is changed into $\psi_2^{(-1)}$ when y describes a certain path.

10. **Further questions.** The definition and the properties of the ψ-functions suggest the investigation of some other connected functions defined by integrals of the form $\int dx/z$ (z an integral of the above differential equation) which I shall try to consider in another paper. On the other hand the results which I have stated in the case of equation (4) have to be extended to the most general equation (1) of type A (see 1) and first, of all, to equation

$$(9) \qquad zz' = 3mz + ax^3 + bx^2 + cx + d$$

for any values of the coefficients m, a, b, c, d.

In this respect, I may remark, that, while it is easy to make this extension, and thus to find *one solution* of the group-problem for the general equation (9), it is more difficult to find the *simplest* solution, which will be quite different for different values of $m, a, \ldots d$. In other words, we always have a group G which we can build up with a limited number of fundamental substitutions, but the choice of the substitutions which it will be most convenient to take as the fundamental ones (and for which

the *fields* called F_1, F_2, ..., etc. will be simplest), will be different for different kinds of equations (9).

One interesting case is a case of which we have an instance by making $m = 1$ in equation (4). For the equation

$$zz' = 3z + 2(x^3 - 1) \tag{10}$$

the number η_1 (equal to $6m(m^3 - 1)$ is 0 and substitution (S_3) disappears. In this case, it is actually possible to build up group G with only the two substitutions (S_1), (S_2), and the inverse substitution, these substitutions being defined in fields similar to those considered in section 7. Equation (10) has, among its solutions, one particular solution which is the polynomial $z = x^2 + x + 1$.

It will be noticed, that, while $\eta_1 = 0$, the number called η_2 (see 4) is not 0 for equation (10). An equation (9) for which both η_1 and η_2 are 0 at the same time would be an equation having, among its solutions, two polynomials, for instance

$$zz' = 3z + 2x(x - 1)(x - 2),$$

which is satisfied by $z = x(x - 1)$ and $z = -(x - 1)(x - 2)$. It is easily proved that such a differential equation can be solved in terms of elliptic functions and does not, therefore, define any new functions.